Higher Derivative Sigma Models

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Goal: Preparing for a lattice effort in HD theories
Logarithmic running couplings differ from literature
Insights for Quadratic Gravity and Asymptotic Safety

Past and present work with Gabriel Menezes
New work with Diego Buccio and Roberto Percacci
Lattice effort with GM, Dean Lee, Alexei Bazarov and Yash Mandlecha

Amherst Center for Fundamental Interactions
Physics at the interface: Energy, Intensity, and Cosmic frontiers
University of Massachusetts Amherst

Quantum gravity
and cosmology
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The non-linear sigma model as a gravity analog:

Gravity

$$g^{\mu\nu} = (g_{\mu\nu})^{-1}$$

$$\mathcal{L} = \sqrt{-g} M_P^2 R + \ldots$$

Sigma model

$$U = e^{i\tau\cdot\pi/v}$$

$$\mathcal{L} = \frac{v^2}{4} Tr(\partial_\mu U^\dagger \partial^\mu U) + \ldots$$

Both form effective field theory at low energy

Both have higher derivative extension
- Quadratic gravity vs HDNLSM (Hasenfratz)

Both have been treated in Asymptotic Safety
- including HDNLSM (Percacci and Zanusso)

But – NLSM is much easier to probe on lattice
- HDNLSM studies the higher derivative aspects of the theory

Outline:

1) **The higher derivative linear sigma model**
   - renormalization without running
   - running without renormalization
   - EFT logic
   - lattice results

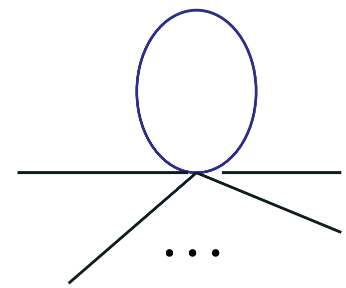
2) **Higher derivative U(1) NLSM**
   - EFT logic
   - non-trivial amplitude calculation
   - lessons on running
   - **new phenomenon**: operator "melting"

3) **The higher derivative SU(N) NLSM**
   - **all** results in literature are changed

4) Comments on UV strong interactions

A Curiosity:

Tadpole diagram with quartic propagators



With UV cutoff (Λ) or IR cutoff (k):

$$I_{tad} = -i \int d^4p \frac{1}{p^4} \sim \log \frac{\Lambda^2}{k^2}$$

With dimensional regularization:

$$I_{tad} = -i \int d^dp \frac{1}{p^4} = 0$$

This is well-known phenomenon and does not lead to any differences in physical reactions

The $\log \Lambda^2/k^2$ dependence disappears in renormalization process

But this leads us astray on running couplings

Calculated by $\Lambda \frac{d}{d\Lambda}$ or $k \frac{d}{d\mathrm{k}}$ (in FRG)

Vanishing of dim. reg. integral shows that this is wrong

Moreover, tadpole integrals also appear in PV reduction of bubbles

What is going wrong?

Tadpole does not involve any external momentum

- no kinematic dependence
- disappears in renormalization process
- measure coupling at any scale, it is the same at another scale

Need to differentiate $\log \Lambda^2$, $\log k^2$, or $\log \mu^2$ from $\log E^2$

1) The Higher Derivative Linear Sigma Model

JFD +
G Menezes

$$\mathcal{L} = \frac{1}{2}\partial_\mu \boldsymbol{\phi} \cdot \partial^\mu \boldsymbol{\phi} - \frac{1}{2M^2}\Box\boldsymbol{\phi} \cdot \Box\boldsymbol{\phi} + \frac{\mu_0^2}{2}\boldsymbol{\phi} \cdot \boldsymbol{\phi} - \frac{\lambda}{4}(\boldsymbol{\phi} \cdot \boldsymbol{\phi})^2$$

Goals here:

A) **not all renormalization is running**

- using $\mu \dfrac{\partial}{\partial\mu}$ is not appropriate here

B) **not all running is renormalization**

- temporary running, but μ, Λ, k independent

C) **EFT interpretation**

D) **lattice simulation** (Jansen, Kuti, Liu)

- theory appears non-perturbatively stable

The Higher Derivative Linear Sigma Model

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \cdot \partial_\mu \phi - \frac{1}{2m^2}\Box\phi \cdot \Box\phi + \frac{\mu_0^2}{2}\phi \cdot \phi - \frac{\lambda}{4}(\phi \cdot \phi)^2$$

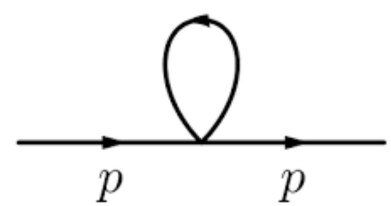
This is renormalizable:

Only divergence is in mass term
- was quadratic, now logarithmic

Comes from tadpole diagram

$$\begin{aligned} \delta\mu_0^2 &= -3i\lambda \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - k^4/m^2} \\ &= -3i\lambda \int \frac{d^4k}{(2\pi)^4} \left[\frac{1}{k^2} - \frac{1}{k^2 - m^2}\right] \end{aligned}$$

Partial fractions



- no dependence on external momenta

$$\delta\mu_0^2 = \frac{3\lambda m^2}{16\pi^2}\left[\frac{1}{\bar{\epsilon}} - \log m^2/\mu^2\right]$$

A) Renormalization without running

$$\mu_0^2|_{phys} = \mu_0^2|_{bare} + \frac{3\lambda m^2}{16\pi^2}\left[\frac{1}{\bar{\epsilon}} - \log m^2/\mu^2\right]$$

Despite $\mu \frac{\partial}{\partial \mu} \delta\mu_0^2 \neq 0$,

this is not a running parameter in physical reactions

- no dependence on external momenta
- measure the same value at any scale

Issue here:

Intrinsic scale m

No sense of mass independent renormalization

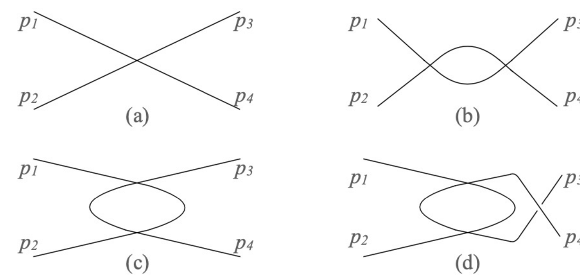
$-\log m^2/\mu^2$ vs $\log q^2/\mu^2$

B) “Running without renormalization” - λ

Slight misnomer

Radiative corrections to λ are finite (and independent of μ, Λ, k)

$$\delta\lambda = -\frac{9\lambda^2}{32\pi^2}\left[\bar{I}_2(q,0,0) - 2\bar{I}_2(q,0,m) + \bar{I}_2(q,m,m)\right]$$



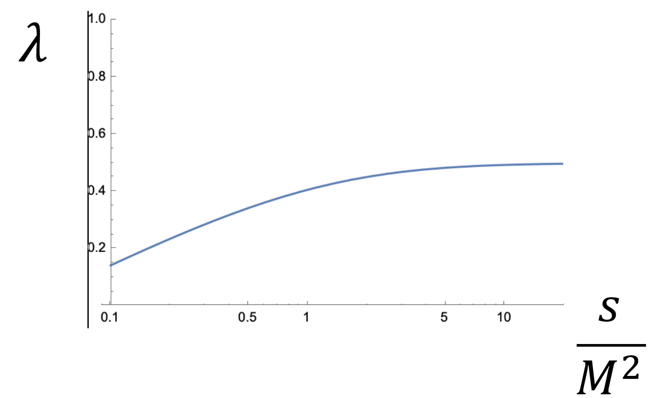
with bubble diagram integral

$$\bar{I}_2(q,m_1,m_2) = \left[\frac{1}{\bar{\epsilon}} - \int_0^1 dx\ \log\left(\frac{xm_1^2 + (1-x)m_2^2 - q^2x(1-x)}{\mu^2}\right)\right]$$

This runs as normal at energies below m (even though $\mu\frac{\partial}{\partial\mu} = 0$)

-measure λ at some renormalization point μ_R

$$\lambda(q) = \lambda(\mu_R) + \frac{9\lambda^2}{32\pi^2}\log(q^2/\mu_R^2)$$



But stops running at high energy

$$\lambda(q) \to \lambda(\infty) = \text{constant}$$

Novel form of Asymptotic Safety

C) EFT understanding

Below m can integrate out heavy ghost
- only residual is light Goldstone field
- EFT theory is normal ϕ^4theory
- normal running

Above m we see full theory
- full theory has cancelling factor of $\log q^2$

D) Lattice simulation

Jansen, Kuti, Liu ~ 1992-94

use third order HD operator – makes theory **finite**

$$\mathcal{L} = -\frac{1}{2}\Phi_\alpha(x)\left(m^2 + \Box + M^{-4}\Box^3\right)\Phi_\alpha(x) - \lambda_0\left(\Phi_\alpha(x)\Phi_\alpha(x)\right)^2,$$

Similar running behavior:

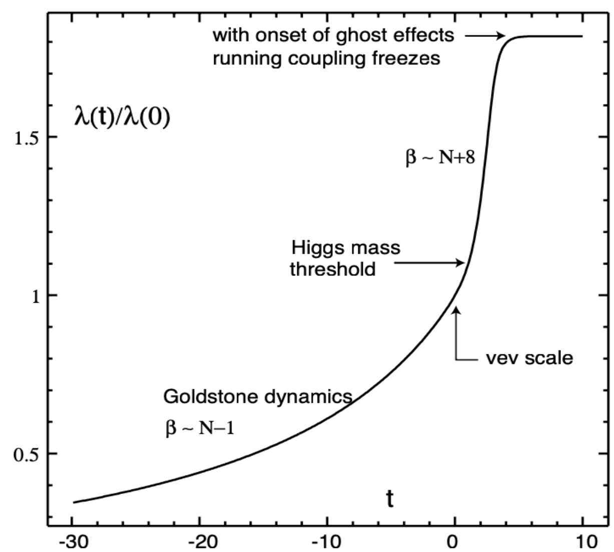


Figure 3. The running coupling constant is plotted from the numerical integration of the one-loop renormalization group equations with input parameters as defined earlier.

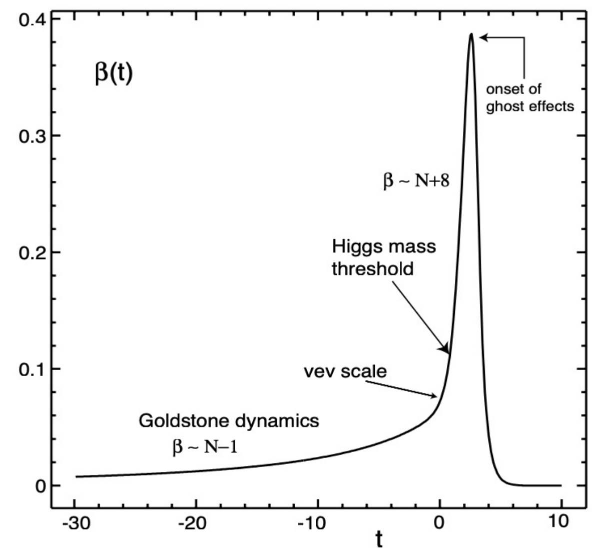


Figure 2. The one-loop β-function is plotted against the logarithmic scale $t = \ln\mu^2/\nu^2$ from the numerical integration of the renormalization group equations. The point $t = 0$ corresponds to $\mu = \nu$ and $t = 0$ on the logarithmic scale. The initial condition $\lambda(0) = 10/24$ is chosen with $M/\nu = 14.3$ which puts the ghost location into the multi-TeV range.

Various results:

- main take-away is **non-perturbative stability**

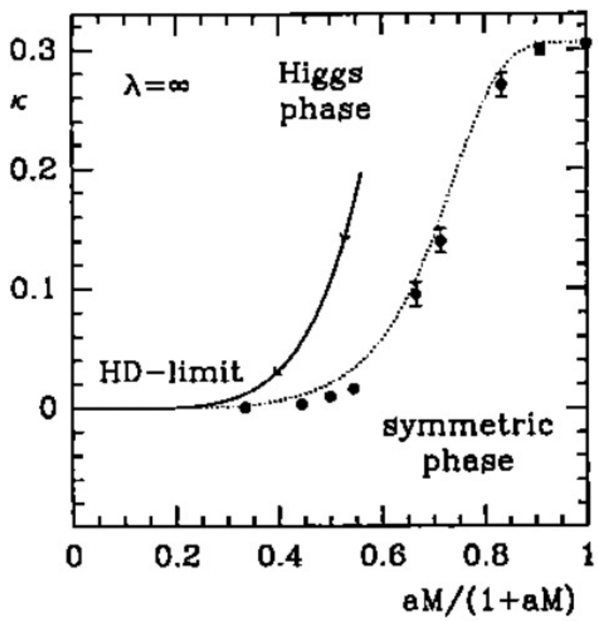


Figure 1. The phase diagram of the lattice model at infinite bare coupling. The dotted line is calculated in the large-N expansion. The solid line displays the fixed M_R/m_H ratio towards the continuum limit of the higher derivative theory.

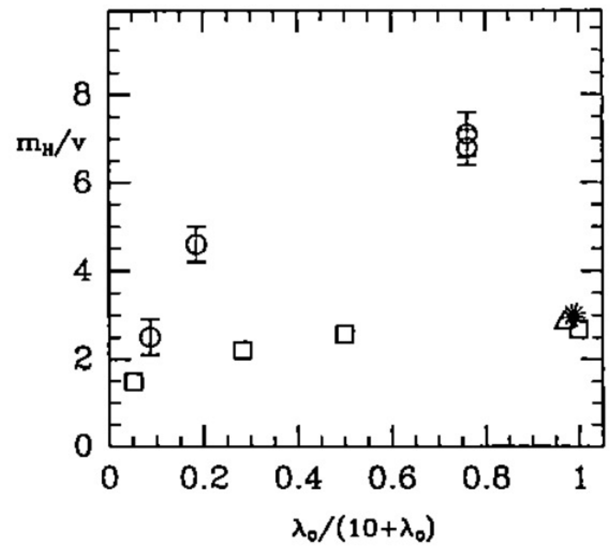


Figure 2. The circles are from our simulation results. They are compared with the simple $O(4)$ model on a hypercubic lattice [2] (squares), with Symanzik improved action on a hypercubic lattice [5] (star), and with dimension six interaction terms added on F_4 lattices [6] (triangle).

Exploring physics of a heavy Higgs

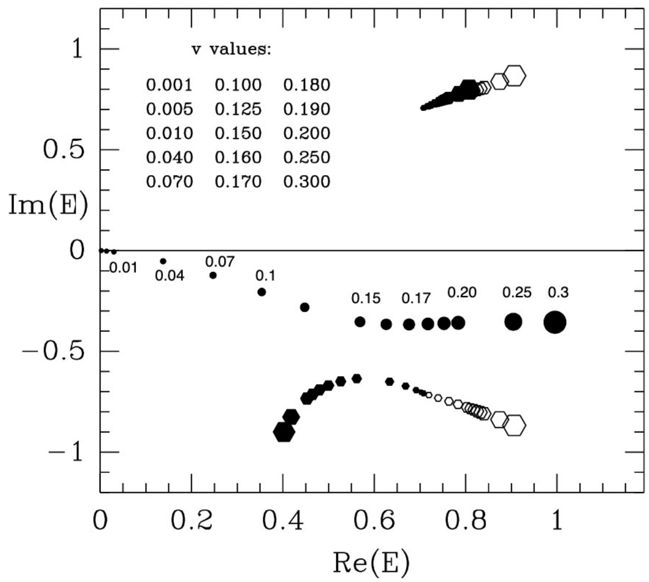


Figure 3.9: The complex poles of the large N Higgs propagator is shown on the first and the second Riemann sheets. The bare coupling constant is set to infinity in this figure. The open hexagonal points represent the ghost pair poles on the first Riemann sheet. The filled hexagonal points are the 'image' of the ghost on the second Riemann sheet. The filled circles are the Higgs poles on the second sheet. The size of the points reflects the different v values.

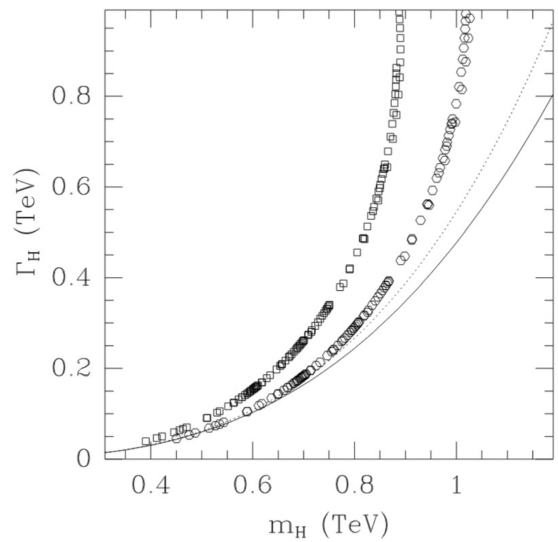


Figure 3.11: The large N result for the width of the Higgs particle as a function of the Higgs mass is shown in the Pauli-Villars higher derivative $O(N)$ theory. The open squares are the naive large N prediction at $N = 4$. The open hexagons are the large N results after the number of decay channels has been corrected. The solid line is the leading order perturbation result and the dashed line is the perturbation result up to the second order. The corrected large N width agrees with the perturbative prediction very well in the weakly interacting regime as it should. The naive large N result overshoots by about 30 to 40 percent.

Buccio, JFD, Percacci
Tseytlin
Holdom

2) The HD U(1) NLSM

- interactions contain derivatives also
- a step closer to quadratic gravity behavior

$$\mathcal{L} = \frac{Z_1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{Z_1}{2m^2}\Box\phi\Box\phi - \frac{Z_1^2 g}{4M^4}(\partial_\mu\phi\partial^\mu\phi)(\partial_\nu\phi\partial^\nu\phi)$$

- *m* and *M* of similar size in ``realistic'' setting
- but we can entertain different sizes
- keep *M* fixed – let *g* potentially run

Renormalizable HD theory, with HD interactions

We have explored full explicit calculations of amplitudes

A) EFT treatment of normal U(1) NLSM

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{g}{M^4}(\partial_\mu\phi\partial^\mu\phi)(\partial_\nu\phi\partial^\nu\phi) + \mathcal{L}_6 + \mathcal{L}_8$$

At one loop, amplitude is of order $\frac{g^2E^8}{M^8}\left[\frac{1}{\epsilon} + \log E^2\right]$

No wavefunction renormalization ($Z_1 = 1$)

Original coupling g is not renormalized (nor any correction)

New operators at order E^8 needed

$$\begin{aligned}
\mathcal{L}_6 &= \frac{g_6}{4M^6}\partial_\mu\phi\partial^\mu\phi\Box\partial_\nu\phi\partial^\nu\phi + \frac{g_6'}{4M^6}\partial_\mu\phi\partial_\nu\phi\Box\partial^\mu\phi\partial^\nu\phi \\
\mathcal{L}_8 &= -\frac{g_8}{4M^8}\partial_\mu\phi\partial^\mu\phi\Box^2\partial_\nu\phi\partial^\nu\phi - \frac{g_8'}{4M^8}\partial_\mu\phi\partial_\nu\phi\Box^2\partial^\mu\phi\partial^\nu\phi
\end{aligned}$$

- g_8 runs (because of $\log E^2$)

The coupling g does not run:

- nor do g_6 couplings
- loops are at order E^8

The low energy matrix element:

$$
\begin{aligned}
\mathcal{M} &= \frac{g}{2M^4}(s^2+t^2+u^2) \\
&+ \frac{g_6}{2M^6}(s^3+t^3+u^3) + \frac{g_6'}{4M^6}(s^2t+s^2u+t^2u+t^2s+u^2s+u^2t) \\
&+ \frac{g_8(\mu_R)}{M^8}(s^4+t^4+u^4) + \frac{g_8'(\mu_R)}{2M^8}(s^2t^2+s^2u^2+t^2u^2) \\
&+ \frac{g^2}{1920\pi^2 M^8}\left[41s^4\log(\frac{-s}{\mu_R^2}) + 41t^4\log(\frac{-t}{\mu_R^2}) + 41u^4\log(\frac{-u}{\mu_R^2})\right. \\
&+ \left. s^2(t^2+u^2)\log(\frac{-s}{\mu_R^2}) + t^2(s^2+u^2)\log(\frac{-t}{\mu_R^2}) + u^2(t^2+s^2)\log(\frac{-u}{\mu_R^2})\right]
\end{aligned}
$$

with

$$\beta_g = 0$$

$$\beta_{g_6} = 0$$

$$\beta_{g_6'} = 0$$

$$\beta_{g_8} = \frac{41g^2}{480\pi^2}$$

$$\beta_{g_8'} = \frac{g^2}{240\pi^2}$$

$$\beta_{Z_1} = 0$$

B) FRG treatment of the Higher Derivative model

- both full AS technology and also at one loop
- Gaussian fixed points in the IR and UV

Buccio
Percacci

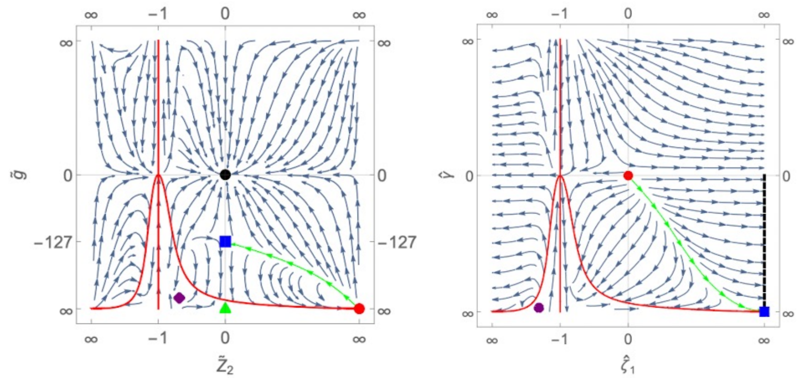


Figure 1: The flow in the chart U_1 (left) and in the chart U_2 (right). The black dot (on the left) and the dahed black line (on the right) mark GFP_1, the red dot marks GFP_2, the blue square marks $NGFP_1$. The separatrix is the green flow line. The continuous red lines are singularities of the flow.

One loop beta functions:

$$\beta_{Z_1} = -\frac{Z_1 + 2k^2/m^2}{16\pi^2(Z_1 + k^2/m^2)^2}\frac{gk^4}{M^4} \quad .$$

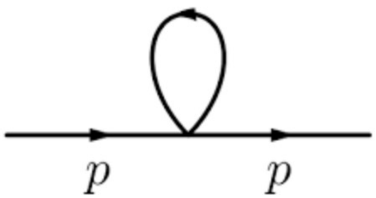
$$\beta_g = \frac{5(Z_1 + 2k^2/m^2)}{32\pi^2(Z_1 + k^2/m^2)^3}\frac{gk^4}{M^4} \cdot$$

C) Explicit one-loop calculation of amplitudes:

- using dim. reg.

Two point function:

- only tadpole diagram



$$i\frac{3}{2}\frac{1}{Z_1}\left(\frac{m}{M}\right)^4 p^2 \frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon}+\log 4\pi-\gamma-\log\frac{m^2}{\mu^2}+\frac{7}{6}+O(\epsilon)\right)$$

Only renormalizes the two-derivative kinetic energy

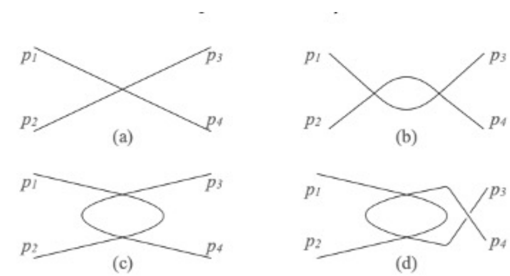
$$\beta_{Z_2}=0$$

But also there is no energy dependence within tadpole

$$\boxed{\beta_{Z_1}=0}$$

Full calculation of the scattering amplitude:

$$
\begin{aligned}
&\frac{5ig^2\left(s^2+t^2+u^2\right)}{64\pi^2 Z_2^2\epsilon}+\frac{ig^2}{5760\pi^2 Z_1^2 Z_2^5}\Bigg\{\\
&+\frac{Z_1^2 Z_2}{s^2}\Big[-6Z_1^2\left(s^2+t^2+u^2\right)+3sZ_1Z_2\left(-31s^2+9\left(t^2+u^2\right)\right)\\
&\qquad +2s^2Z_2^2\left((352-195\gamma)s^2-(15\gamma-37)\left(t^2+u^2\right)\right)\Big]\\
&+6s^{-1/2}Z_2^{5/2}\sqrt{4Z_1-sZ_2}\,[16Z_1^2(6s^2+t^2+u^2)\\
&\qquad -8sZ_1Z_2(16s^2+t^2+u^2)+s^2Z_2^2(41s^2+t^2+u^2)]\operatorname{arccot}\sqrt{\frac{4Z_1}{sZ_2}-1}\\
&+\frac{Z_1^2 Z_2}{t^2}\Big[-6Z_1^2\left(s^2+t^2+u^2\right)+3tZ_1Z_2\left(-31t^2+9(s^2+u^2)\right)\\
&\qquad +2t^2Z_2^2\left((352-195\gamma)t^2-(15\gamma-37)\left(s^2+u^2\right)\right)\Big]\\
&+6t^{-1/2}Z_2^{5/2}\sqrt{4Z_1-tZ_2}\,[16Z_1^2(s^2+6t^2+u^2)\\
&\qquad -8tZ_1Z_2(s^2+16t^2+u^2)+t^2Z_2^2(s^2+41t^2+u^2)]\operatorname{arccot}\sqrt{\frac{4Z_1}{tZ_2}-1}\\
&\frac{Z_1^2 Z_2}{u^2}\Big[-6Z_1^2\left(s^2+t^2+u^2\right)+3uZ_1Z_2\left(-31u^2+9\left(s^2+t^2\right)\right)\\
&\qquad +2u^2Z_2^2\left((352-195\gamma)u^2-(15\gamma-37)\left(s^2+t^2\right)\right)\Big]\\
&+6u^{-1/2}Z_2^{5/2}\sqrt{4Z_1-uZ_2}\,[16Z_1^2(s^2+t^2+6u^2)\\
&\qquad -8uZ_1Z_2(s^2+t^2+16u^2)+u^2Z_2^2(s^2+t^2+41u^2)]\operatorname{arccot}\sqrt{\frac{4Z_1}{uZ_2}-1}\\
&+3s^2Z_2^5\left(41s^2+t^2+u^2\right)\log\left(-\frac{Z_1}{sZ_2}\right)\\
&+3t^2Z_2^5\left(s^2+41t^2+u^2\right)\log\left(-\frac{Z_1}{tZ_2}\right)\\
&+3u^2Z_2^5\left(s^2+t^2+41u^2\right)\log\left(-\frac{Z_1}{uZ_2}\right)\\
&+\frac{6(-uZ_2+Z_1)^3}{u^3}\log\left(\frac{Z_1}{Z_1-uZ_2}\right)\left[Z_1^2\left(s^2+t^2+u^2\right)-2uZ_1Z_2\left(s^2+t^2-9u^2\right)+u^2Z_2^2\left(s^2+t^2+41u^2\right)\right]\\
&+\frac{6(-tZ_2+Z_1)^3}{t^3}\log\left(\frac{Z_1}{Z_1-tZ_2}\right)\left[Z_1^2\left(s^2+t^2+u^2\right)-2tZ_1Z_2\left(s^2-9t^2+u^2\right)+t^2Z_2^2\left(s^2+41t^2+u^2\right)\right]\\
&+\frac{6(-sZ_2+Z_1)^3}{s^3}\log\left(\frac{Z_1}{Z_1-sZ_2}\right)\left[Z_1^2\left(s^2+t^2+u^2\right)-2sZ_1Z_2\left(-9s^2+t^2+u^2\right)+s^2Z_2^2\left(41s^2+t^2+u^2\right)\right]\\
&+450Z_1^2Z_2^3\left(s^2+t^2+u^2\right)\log\left(\frac{4\pi\mu^2Z_2}{Z_1}\right)\Bigg\}
\end{aligned}
\tag{3.25}
$$



Understanding the scattering amplitude

Two energy regions with different behavior:

1) Low energy ($E < m$)
   - massive particle not dynamically active
   - integrating out and forming EFT

2) High energy ($E > m$)
   - all d.o.f. are active
   - simplifies amplitudes considerably

The beta functions can be different in these two regions

Low energy:

Renormalize *g* here: (“renormalize without running”)

$$g = g_0 - \frac{5g^2m^4}{32\pi^2M^4}\left[\frac{1}{\epsilon} + \gamma - \log\left(\frac{4\pi\mu^2}{m^2}\right) + \frac{11}{30}\right]$$

So $\beta_g = 0$ at low energy

Residual matches EFT exactly with

$$\begin{aligned} g &= g \\ g_6 &= -\frac{53g^2m^2}{384\pi^2M^2} \\ g_6' &= -\frac{7g^2m^2}{516\pi^2M^2} \\ g_8(\mu_R) &= \frac{79g^2}{1920\pi^2} + \frac{41g^2}{960\pi^2}\log\frac{\mu_R^2}{m^2} \\ g_8'(\mu)_R &= \frac{3g^2}{320\pi^2} + \frac{g^2}{480\pi^2}\log\frac{\mu_R^2}{m^2} \quad . \end{aligned}$$

But g_8 is finite here – “running without renormalization”

Recall the EFT matrix element:

The low energy matrix element:

$$
\begin{aligned}
\mathcal{M} &= \frac{g}{2M^4}(s^2+t^2+u^2) \\
&+ \frac{g_6}{2M^6}(s^3+t^3+u^3) + \frac{g_6'}{4M^6}(s^2t+s^2u+t^2u+t^2s+u^2s+u^2t) \\
&+ \frac{g_8(\mu_R)}{M^8}(s^4+t^4+u^4) + \frac{g_8'(\mu_R)}{2M^8}(s^2t^2+s^2u^2+t^2u^2) \\
&+ \frac{g^2}{1920\pi^2 M^8}\left[41s^4\log(\frac{-s}{\mu_R^2}) + 41t^4\log(\frac{-t}{\mu_R^2}) + 41u^4\log(\frac{-u}{\mu_R^2})\right. \\
&+ \left. s^2(t^2+u^2)\log(\frac{-s}{\mu_R^2}) + t^2(s^2+u^2)\log(\frac{-t}{\mu_R^2}) + u^2(t^2+s^2)\log(\frac{-u}{\mu_R^2})\right]
\end{aligned}
$$

with

$$
\begin{aligned}
\beta_g &= 0 \\
\beta_{g_6} &= 0 \\
\beta_{g_6'} &= 0
\end{aligned}
\qquad
\begin{aligned}
\beta_{g_8} &= \frac{41g^2}{480\pi^2} \\
\beta_{g_8'} &= \frac{g^2}{240\pi^2}
\end{aligned}
\qquad
\beta_{Z_1} = 0
$$

EFT understanding

At energies below m can integrate out heavy ghost
- only dynamical residual is light Goldstone field

Matches EFT description

- can be described by measurable local parameters $g, g_8, \ldots$
- no running in g
- running in g_8 at low energy
- as expected, also new local operator at E^6
    - coupling g_6 predicted by the full theory

<u>High energy:</u>

In terms of already renormalized coupling:

$$\mathcal{M} = \frac{g}{2M^4}\left[1+\frac{13gm^4}{128\pi^2 M^4}\right](s^2+t^2+u^2)$$
$$+ \frac{g^2m^4}{192\pi^2M^8}\left[\log\left(\frac{-s}{m^2}\right)(13s^2+t^2+u^2)+\log\left(\frac{-t}{m^2}\right)(s^2+13t^2+u^2)+\log\left(\frac{-u}{m^2}\right)(s^2+t^2+13u^2)\right]$$

1) **Higher energy dependence has melted away**

- E^8 and E^6 dependence is no longer present
- multiple cancellations
- bubble diagrams

$$I_2(m_1,\ m_2,\ q^2) = \frac{1}{16\pi^2}\left[\frac{1}{\epsilon}+\gamma-\log 4\pi-\int_0^1 dx\log\left(\frac{xm_1^2+(1-x)m_2^2-q^2x(1-x)}{\mu^2}\right)\right] .$$

at high energy:

$$[I_2(0,0,s)-2I_2(0,m,s)+I_2(m,m,s)] \sim \frac{m^2}{s}\log\frac{-s}{m^2}+...$$

New phenomenon: (AFAIK)

Operator "melting"

- Need E^6, E^8 operators for $E < m$

- But effects disappear for $E > m$

2) **Can define running coupling here:**

- in terms of already renormalized coupling

$$\bar{g}(\mu_R) = g + \frac{5g^2m^4}{32\pi^2M^4}\left[\log\left(\frac{\mu_R^2}{m^2}\right) + \frac{13}{20}\right]$$

Removes large logarithms

$$\begin{aligned}\mathcal{M} &= \frac{\bar{g}(\mu_R)}{2M^4}(s^2+t^2+u^2) \\ &+ \frac{g^2m^4}{192\pi^2M^8}\left[\log\left(\frac{-s}{\mu_R^2}\right)(13s^2+t^2+u^2) + \log\left(\frac{-t}{\mu_R^2}\right)(s^2+13t^2+u^2) + \log\left(\frac{-u}{\mu_R^2}\right)(s^2+t^2+13u^2)\right]\end{aligned}$$

Beta function here **agrees** with asymptotic form of FRG

$$\beta_{\bar{g}} = \frac{5\bar{g}^2m^4}{16\pi^2M^4}$$

$$\beta_g = \frac{5(Z_1+2k^2/m^2)}{32\pi^2(Z_1+k^2/m^2)^3}\frac{g^2k^4}{M^4} \to \frac{5g^2m^4}{16\pi^2M^4}, \qquad k >> m$$

Understanding non-running (LE) vs running (HE)

Loop diagram in this theory

$$\begin{aligned} I_{\mu\nu\alpha\beta} &= m^4 \int \frac{d^dp}{(2\pi)^d} \frac{p_\mu p_\nu (p-q)_\alpha (p-q)_\beta}{[m^2p^2 - p^4][(m^2(p-q)^2 - (p-q)^4]} \\ &= F(q^2)(\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}) \\ &+ \text{ order q terms} \end{aligned}$$

Only divergence is in F

$$\begin{aligned} \eta^{\mu\nu}\eta^{\alpha\beta} I_{\mu\nu\alpha\beta} &= m^4 \int \frac{d^dp}{(2\pi)^d} \frac{1}{[m^2 - p^2][(m^2 - (p-q)^2]} \\ &= = m^4 I_2(m,m,q) = d(d+2)F + ... \end{aligned}$$

$$I_2(m_1,\ m_2,\ q^2) = \frac{1}{16\pi^2}\left[\frac{1}{\epsilon} + \gamma - \log 4\pi - \int_0^1 dx \log\left(\frac{xm_1^2 + (1-x)m_2^2 - q^2x(1-x)}{\mu^2}\right)\right] .$$

Low energy $\log m^2/\bar{\mu}^2$ - no running

High energy $\log \bar{q}^2/\mu^2$ - running

<u>This is standard behavior</u>

Example top quark contribution to running of **α**
- at low energy

$$\Pi(q^2) = \frac{\alpha}{3\pi}\left[\frac{1}{\epsilon} - \gamma + \log 4\pi - \log\frac{m_t^2}{\mu^2} + \frac{q^2}{5m_t^2} + ...\right]$$

- divergence or scale dependence does not imply running

- contributes to running only past top threshold

In these theories there is a mass parameter and
a mass threshold

3) Higher Derivative Nonlinear Sigma Model

P. Hasenfratz (1989)

$$S = (1/f^2)(c_0 S_0 + S_1 + c_2 S_2 + c_3 S_3 + c_4 S_4 + c_5 S_5)$$

This is closest equivalent to quadratic gravity

- non-linear
- lowest order is non-perturbative EFT
- Hasenfratz calculates it as asymptotically free (one loop)

$$\frac{d}{dt} f^2 = -\frac{N}{32\pi^2} f^4$$

- should be able to calculated on lattice
- also comparison to Asymptotic Safety (Percacci, Zanusso)

Higher Derivative Nonlinear Sigma Model

P. Hasenfratz (1989)

$$U(x) = e^{if\pi^a(x)t^a}$$

$$S = (1/f^2)(c_0 S_0 + S_1 + c_2 S_2 + c_3 S_3 + c_4 S_4 + c_5 S_5)$$

$$S_0 = \int d^4x \, \mathrm{Tr}\left(U^{-1}(x)\,\partial_\mu U(x)\,U^{-1}(x)\,\partial_\mu U(x)\right)$$

← Lowest order

$$S_1 = \tfrac{1}{2}\int \mathrm{Tr}(\partial_\mu A_\mu\,\partial_\nu A_\nu + \partial_\mu A_\nu\,\partial_\mu A_\nu),$$

← Higher deriv. in propagator

$$S_2 = -\tfrac{1}{2}\int \mathrm{Tr}(\partial_\mu A_\mu\,\partial_\nu A_\nu - \partial_\mu A_\nu\,\partial_\mu A_\nu),$$

$$A_\mu(x) = U^{-1}(x)\,\partial_\mu U(x)$$

$$S_3 = -\tfrac{1}{2}\int \mathrm{Tr}(A_\mu A_\mu A_\nu A_\nu + A_\mu A_\nu A_\mu A_\nu),$$

$$S_4 = -\int \mathrm{Tr}(A_\mu A_\mu)\,\mathrm{Tr}(A_\nu A_\nu),$$

$$S_5 = -\int \mathrm{Tr}(A_\mu A_\nu)\,\mathrm{Tr}(A_\mu A_\nu),$$

← Invariant interactions starting at 4 pions (S_2, S_3, S_4, S_5)

EFT for $E < m$

Gasser Leutwyler
Bijnens, Lu

Usual two derivative theory:

$$S_{\text{quad}} = \frac{1}{2c_0}\int d^4x\, \pi^a \mathcal{D}^{ab}\pi^b$$
$$\mathcal{D}^{ab} = \delta^{ab}D^2 + \hat{\sigma}^{ab}.$$

Here the E^4 term do run in this limit

$$a_2 = \frac{1}{12}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \frac{1}{2}\hat{\sigma}^2.$$

Beta functions here are pure numbers – independent of couplings

$$\bar{\beta}_1 = \frac{1}{4}$$
$$\bar{\beta}_2 = N$$
$$\bar{\beta}_3 = \frac{1}{2}$$
$$\bar{\beta}_4 = 0 =$$

These do not match Hasenfratz nor FRG results

Full evaluation for *E > m*:

G Menezes

- background field method

$$S_{\text{quad}} = \frac{1}{2}\int d^4x\, \pi^a \mathcal{D}^{ab}\pi^b$$

$$\mathcal{D}^{ab} = -\frac{1}{f^2}\delta^{ab} D_\mu D^\mu D_\nu D^\nu + \mathcal{B}^{ab}_{(\mu\nu)} D^\nu D^\mu + \mathcal{C}^{ab}_\mu D^\mu + \mathcal{E}^{ab}$$

Divergent heat kernel coefficient

$$a_2 = \frac{1}{6}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \frac{f^4}{24}\mathcal{B}_{(\mu\nu)}\mathcal{B}^{\mu\nu} + \frac{f^4}{48}\mathcal{B}^2 + f^2\mathcal{E}$$

Barvinsky
Vilkovisky

But need to remove tadpole effects

$$\begin{aligned}
\beta_1 &= \frac{f^4}{24}b_1 + \frac{f^4}{48}b_5 - \frac{1}{2} + \frac{4Nf^2}{\alpha_4^2}\\
\beta_2 &= \frac{f^4}{24}b_2 + \frac{f^4}{48}b_6 - 2N + \frac{10Nf^2}{\alpha_3^2} - \frac{16f^2}{\alpha_4^2} + \frac{8f^2}{\alpha_5^2}\\
\beta_3 &= \frac{f^4}{24}b_3 + \frac{f^4}{48}b_7 - 1 + \frac{4Nf^2}{\alpha_5^2}\\
\beta_4 &= \frac{f^4}{24}b_4 + \frac{f^4}{48}b_8 + \frac{6Nf^2}{\alpha_3^2} + \frac{16f^2}{\alpha_4^2} - \frac{8f^2}{\alpha_5^2}.
\end{aligned}$$

$$\longrightarrow$$

$$\begin{aligned}
\beta_1 &= \frac{f^4}{24}b_1 + \frac{f^4}{48}b_5\\
\beta_2 &= \frac{f^4}{24}b_2 + \frac{f^4}{48}b_6\\
\beta_3 &= \frac{f^4}{24}b_3 + \frac{f^4}{48}b_7\\
\beta_4 &= \frac{f^4}{24}b_4 + \frac{f^4}{48}b_8.
\end{aligned}$$

$$\mu\frac{df}{d\mu} = -\frac{N}{64\pi^2}f^3 = \beta_f$$

$$\beta_f = 0.$$

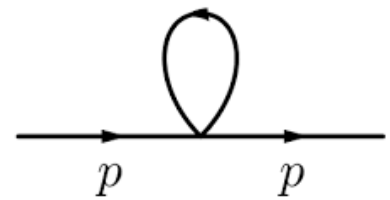
This yields different results than cutoff or FRG

Specific example: Renormalization of basic coupling *f*

- appears in two-point function - renormalization of propagator

$$D(q) = \frac{1}{M^2 q^2 - q^4/f^2}$$

- HDNLSM conserves parity
    - only 2, 4 , 6, …. particle vertices occur
- At one loop, this then requires tadpole diagram



This is again a tadpole diagram

Hasenfratz gets running from $\Lambda \frac{\partial}{\partial \Lambda}$ (or $\mu \frac{\partial}{\partial \mu}$)

FRG treatment uses $k \frac{\partial}{\partial k}$

$$\frac{d}{dt} f^2 = -\frac{N}{32\pi^2} f^4$$

But again there is no real running **at any energy**

$$\beta_f = 0.$$

Comparison with FRG

Never see power law running in amplitudes
- new operators with different signs, magnitudes

Logarithmic running is unreliable (some right, some wrong)

$$\begin{aligned}\beta_g &= 0 \qquad (E<m)\\ &= \frac{5g^2m^4}{16\pi^2M^4} \qquad (E>m)\end{aligned}$$

vs

$$\begin{aligned}\beta_g &= \frac{5g^2k^4}{32\pi^2M^4} \qquad (E<m)\\ &= \frac{5g^2m^4}{16\pi^2M^4} \qquad (E>m)\end{aligned}$$

$$\beta_{Z_1} = 0 \qquad (\text{all } E)$$

vs

$$\begin{aligned}\beta_{Z_1} &= \frac{3gm^4}{16\pi^2M^4} \qquad (E<m)\\ &= \frac{gm^2k^2}{M^4} \qquad (E>m)\end{aligned}$$

$$\beta_f = 0 \qquad (\text{all } E)$$

$$\beta_{f^2} = -\frac{Nf^4}{32\pi^2}$$

And all the β_{α_i} are not correct

Implications for gravity:

Quadratic gravity

- EFT to full theory transition
- running of couplings needs to be redone
- will asymptotic freedom of f_{C^2} survive?

Asymptotic Safety

- power law running not seen in amplitudes
- logarithmic running also may not be physical
- EFT to full theory transition
- what does AS running imply for the real world?

But also:

μ dependence does not always imply running

Example: Cosmological Constant does not run

$$\delta\Lambda = -\frac{m^4}{32\pi^2}\left[\frac{1}{\epsilon} - \gamma + \log(4\pi) + \log\frac{\mu^2}{m^2} + \frac{3}{2}\right]$$

Food for thought

1) Does running even matter in these theories?

- we have

$$\mathcal{M} \sim g(\mu)\left(s^2 + t^2 + u^2\right)$$

- even if $g \rightarrow 0$ logarithmically
  $s^2 + t^2 + u^2 \rightarrow \infty$ faster

Holdom objection here

- strong interaction for amplitudes at high energy

2) **Is it a problem to have strong gravity?**

- QCD is renormalizable but strong at low E
- perhaps gravity is renormalizable but strong at high E
- requires alternate understanding

  QCD strings -> gravity strings?

Summary:

Various flavors of renormalization group are **inequivalent**
- using physical amplitudes reveals physical running

Previously: power-law, Λ^2, Λ^4 are not running effects in amplitudes

Here: **Logarithmic running with HD is subtle**
- "renormalization without running"
- " running without renormalization"
- using EFT region for renormalization – runs like EFT there
- transition in logarithmic running at ghost mass
  - running to no running (HDLSM)
  - no running to running (HD shift invariant)
  - no running at any scales (HDNLSM)

Disagrees with some of QQG and FRG literature

Strongly interacting region
- with derivative interactions overwhelms logarithmic running

Probably need numerical method to sort this out