# Effective quantum gravity corrections to $U(r)$ in the consistent framework

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**Partially based on**

*T. de Paula Netto, L. Modesto and I.Sh., EPJC; arXiv: 2110.14263.*

*Tibério de Paula Netto, Lectures on effective quantum gravity, unpublished*

**Special thanks for discussions:** *Roberto Percacci and Josif Frenkel*

# Further reading

## Textbook-level introduction to QG:

*I.L. Buchbinder and I.L. Shapiro, Introduction to Quantum Field Theory with Applications to Quantum Gravity (Oxford University Press, 2021).*

## Initial reading on effective QG:

*J.F. Donoghue, Phys. Rev. Lett. 72 (1994) 2996, gr-qc/9310024; Phys. Rev. D50 (1994) 3874, gr-qc/9405057;* $\exists$ *many reviews, etc.*

*C.P. Burgess, Living Rev. Rel. 7 (2004) 5, gr-qc/0311082.*

## Many interesting introductory-level reviews in the

*Section "Effective Quantum Gravity" edited by C. Burgess and J. Donoghue of the "Handbook of Quantum Gravity" (Editors C. Bambi, L. Modesto and I.L. Shapiro, Springer Singapore, expected in 2023).*

- **Publications with the approach described below.**

**Original paper:** *[DM] D.A.R. Dalvit and F.D. Mazzitelli, PRD56 (1997) 7779, hep-th/9708102.*

**General discussion:**

*I.Sh., Polemic notes on IR perturbative quantum gravity, Int. J. Mod. Phys.* **A24** *(2009) 1557, arXiv:0812.3521.*

**Calculation of effective QG corrections to Newton potential:**

*Tibério de Paula Netto, Leonardo Modesto, and I.Sh., Eur. Phys. J. C82 (2022) 160, arXiv: 2110.14263.*

- **Quantum GR is a universal IR theory of QG?**

*Wagno Cesar e Silva & I.Sh., Effective approach to the Antoniadis-Mottola model: quantum decoupling of the higher derivative terms, arXiv: 2301.13291.*

# I. Effective approach.

**The effective approach is the cornerstone of the application of QFT to particle physics and maybe to all modern physics.**

**This approach explains why we do not care about fundamental UV physical theory when dealing with low-energy phenomena.**

**For example, when we perform calculations of atomic spectra there is almost no need to care about what is going on in the atomic nuclei and absolutely no need to care about what is going on at the level of quarks inside the nuclei.**

**The reason is that the energy scale of the two types of phenomena is very much different. Low-energy experiments may be not sensible to the high energy (UV) interactions.**

**One can find this way of thinking even starting from Classical Mechanics, but in QFT it becomes a practical instrument.**

**Technically, the effective approach assumes separation of the low-energy (IR) quantum effects from the UV sector.**

**The idea was introduced by S. Weinberg in 70-s and became a general QFT framework.**

**We can say that we have to either look for the "theory of everything" or try to restrict our attention to something.**

**And in the last case there are always some parts of Nature which are intentionally left behind because they do not belong to the given set of phenomena.**

**The classical examples are Fermi theory of weak interactions, NJL model of quarks, etc.**

**Another example is MSM which is extremely successful theory of Particle Physics (maybe even too much!), regardless we do know its "UV completion" and even whether there is one.**

**Typically, in the effective approach, we have at least two different scales and in the IR one can ignore the effects of the massive field because they are suppressed by a large mass parameter.**

**Another possibility is to use the fact that the UV divergences are always local and, in case of massive degrees of freedom, they decouple in the IR.**

**The last means there is no non-local form factor in the IR. Thus, the link between the UV divergences and the non-local form factor (usually logarithmic) disappears in the IR.**

**This means, we can use hierarchy of scales and separate only the quantum effects of the light (or massless) fields, assuming only these effects are relevant in the IR, i.e., for the long-distance interactions.**

**From this perspective, quantum gravity is (as J. Donoghue wrote in some review) a perfect theory for applying effective approach.**

# II. Effective low-energy quantum gravity.

**It looks natural to use it for QG, where we actually meet two very different energy scales:** $M_P \approx 10^{19}\, GeV$ **vs the mass of the graviton, which is zero.**

**The global IR regulator comes from cosmology, but its energy scale is related to the size of the Universe, i.e., it is related to Hubble parameter,** $\mu_c \sim H_0 \approx 10^{-42}\, GeV \approx 10^{-60}\, M_P$ **.**

**This is certainly a maximal possible separation between the UV and IR scales and we can expect that this theory will be a perfect field of application of the effective approach.**

**In theory, one can calculate all IR quantities which are potentially relevant for establishing universal features of quantum gravity.**

**There are many different directions in effective QG (see the "Handbook"), but the main traditional object of interest is the quantum corrected Newtonian potential of gravitational attraction between two massive point-like bodies.**

## Standard approach to effective QG

*J.F. Donoghue, Phys.Rev.Lett. 72 (1994); PRD 50 (1994).*

**Correcting first set of mistakes:**

*H.W. Hamber, S. Liu, Phys. Lett. B357 (1995) 51;*
*I.J. Muzinich, S. Vokos, Phys. Rev. D52 (1995) 3472;*
*A.A. Akhundov, S. Bellucci, A. Shiekh, Phys. Lett. B395 (1997) 16.*

**Providing gauge-fixing invariant contributions:**

[DM] *D.A.R. Dalvit and F.D. Mazzitelli, PRD 56 (1997) 7779.*
*J. A. Helayel-Neto, A. Penna-Firme, I.Sh., JHEP 0001 (2000) 009.*

**Should we include all the diagrams?**

*I. Sh., Polemic notes on IR perturbative QG, arXiv:0812.3521.*

**Conflicting results:**

*N.E.J. Bjerrum-Bohr, J.F. Donoghue, B.R. Holstein, PRD67 (2003);*
*I.B. Khriplovich, G.G. Kirilin, J.Exp.Theor.Phys. 95 (2002).*

**Earlier calculation:**

*Y. Iwasaki, Prog. Theor. Phys. 46 (1971) 1587.*

## The standard effective QG starts as usual QG/GR.

*J.F. Donoghue, Phys. Rev. Lett. 72 (1994) 2996, gr-qc/9310024; Phys. Rev. D50 (1994) 3874, gr-qc/9405057; ...* **there are many reviews and lectures.**

$$S_t \,=\, S_{EH} + S_{gf} + S_{ghost} + S_{matter}\,,$$

**where**

$$S_{EH} = -\frac{1}{\kappa^2}\int d^4x\sqrt{-g}\,R\,,$$

$$S_{gf} = \frac{1}{\alpha}\int d^4x\sqrt{-g}\,\chi_\mu\,\chi^\mu\,,\qquad \chi_\mu = \partial_\lambda\, h^\lambda_\mu - \beta\,\partial_\mu h^\lambda_\lambda\,;$$

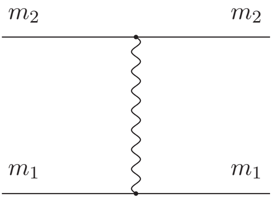
$$S_{ghost} = \int d^4x\sqrt{-g}\,\bar{C}_\mu\,\frac{\delta\chi^\mu}{\delta h_{\rho\sigma}}\,R^{\alpha}_{\ \cdot\,\rho\sigma}\,C_\alpha\,,$$

$$S_{matter} = \int d^4x\sqrt{-g}\,\left\{\frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi \,-\, \frac{1}{2}\,m^2\phi^2\right\}\,.$$

**The assumed purpose is to derive the (quantum) gravitational interaction between two scalars, of masses $m_1$ and $m_2$.**

# At the classical level the situation is relatively simple

**The tree-level scattering amplitude has the form**




$$T(q) \,=\, 4\pi\, \frac{G\, m_1\, m_2}{\vec{q}^{\,2}}\,.$$

**In the static limit** $q^0 = 0$ **and** $q^2 = -\vec{q}^{\,2}$**. After Fourier transform**

$$\int \frac{1}{\vec{q}^{\,2}}\, e^{i\vec{q}\cdot\vec{r}}\, d^3q \,=\, \frac{1}{4\pi r}\,,$$

**we arrive at the Newton potential**

$$V(r) \,=\, -\,\frac{G\, m_1 m_2}{r},$$

**which is the tree-level approximation to the potential for the interaction between two static sources.**

**One-graviton exchange between the two masses gives Newton law in the IR limit. What are the IR quantum corrections?**

**At the one-loop level there are two types of diagrams and two types of IR-relevant contributions.**

## I. P-type terms.

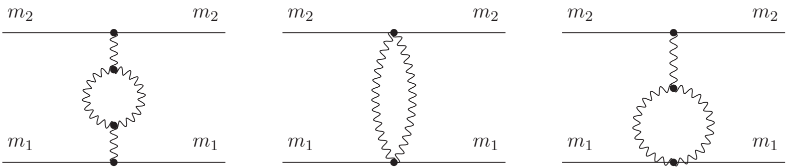
$$\int \frac{d^3q}{(2\pi)^3}\, e^{-i\vec{q}\cdot\vec{r}}\, \frac{1}{\sqrt{\vec{q}^2}} \;=\; \frac{1}{2\pi^2\, r^2}\,.$$

## II. L-type terms.

$$\int \frac{d^3q}{(2\pi)^3}\, e^{-i\vec{q}\cdot\vec{r}}\, \ln \vec{q}^2 \;=\; -\,\frac{1}{2\pi^2\, r^3}\,.$$

**Phenomenologically, the P-type terms look much more interesting. These terms are as the first post-Newtonian approximation and hence we have to expect that QG will reproduce the classical GR result, e.g., the precession of the perihelion of Mercury. And if they do not?**

# I. Graphs with massless (gravitational) internal lines



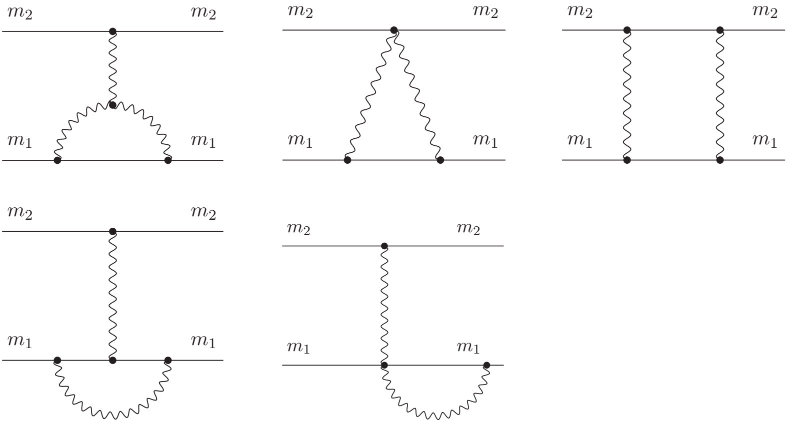

**Those contribute only to the L-type terms.**

**These diagrams correspond to the path integral over metric, while the massive scalar field is an external classical source.**

**From the QFT viewpoint, this means we have to quantize only the metric and then there is no big difference how we describe the matter sources.**

**Also, we can use functional methods instead of diagrams.**

# II. Graphs with both massless and massive (scalar field) internal lines




**These diagrams contribute to both L-type and P-type terms.**

**What means we include these diagrams?**

# Classical or quantum?

**Polemic Note:** *I.Sh., IJMPA; arXiv:0812.3521 [hep-th].*

**Massive scalars ...**
**What about fermions and, finally, macroscopic bodies?**

**The macroscopic bodies which take part in the relevant gravitational interactions are not made from a scalar field.**

**In reality, they do consist from a baryonic matter, that means interacting protons, neutrons and electrons.**

**These particles are not elementary (except electron) and none of them may be properly described by a scalar field.**

**Of course, nucleons consist from quarks and gluons, so one may think to replace the scalar field by the spinor one and try to obtain the quantum gravity corrections taking, e.g., mixed graviton-quark diagrams.**

**However, this way to describe barionic matter would not be consistent because quarks are not free particles.**

**One of the manifestations of this is that the total mass of the $u$, $\bar{u}$ and $d$ quarks is much smaller than the mass of the proton. Calculating even the tree-level diagrams with quarks we have no chance to get a correct result.**

**Another argument is that the Casimir effect corresponds to the quantization of fields, not of the classical sources. And the results fit experimental data pretty well, up to our knowledge.**

**Finally, we arrive at the conclusion that the "correct" set of diagrams includes only $L$-type. From the phenomenological side, this means we are far from observing relevant effects of QG at low energies.**

**Restricting the consideration by the $L$-type diagrams, can we arrive at the well-defined quantum corrections?**

**In particular, can we provide the independence of quantum contributions from the gauge fixing parameters $\alpha$ and $\beta$ ?**

$$S_{gf} = \frac{1}{\alpha}\int d^4x\sqrt{-g}\,\chi_\mu\,\chi^\mu, \quad \chi_\mu = \nabla_\lambda\, h^\lambda_\mu - \beta\,\nabla_\mu h^\lambda_\lambda\,.$$

**And on the choice of parametrization of the quantum metric**

$$\begin{aligned} g_{\mu\nu} \quad \longrightarrow \quad g'_{\mu\nu} \;=\; & g_{\mu\nu} + \kappa(\gamma_1\phi_{\mu\nu} + \gamma_2\phi g_{\mu\nu}) \\ & + \kappa^2(\gamma_3\phi_{\mu\rho}\phi^\rho_\nu + \gamma_4 g_{\mu\nu}\phi_{\rho\sigma}\phi^{\rho\sigma} + \gamma_5\phi\phi_{\mu\nu} + \gamma_6 g_{\mu\nu}\phi^2)\,. \end{aligned}$$

**The consistency of a physical results requires their independence on the parameters $\alpha,\,\beta$ and $\gamma_k$.**

**The unique consistent scheme of the effective QG calculation satisfying this condition, has been developed in**

*[DM] D.A.R. Dalvit, F.D. Mazzitelli, hep-th/9708102, PRD.*

**Here we follow (see further references therein)**

*L. Modesto, Tibério P. Netto, I.Sh., arXiv:1412.0740, JHEP.*

**Consider effective QG interacting with an arbitrary point-like matter particle, not a scalar field.**

$$S = -\frac{1}{\kappa^2}\int d^4x\sqrt{-g}\,R + S_{\mathrm{sources}},$$

**Later on, we restrict our attention by the action of a point-like particle $S_{\mathrm{sources}} = S_M,\ S_m,$ etc., with coordinates $y^\mu$**

$$S_M = -M\int ds = -M\int\sqrt{g_{\mu\nu}\,dy^\mu dy^\nu}.$$

**Furthermore, in the source sector, we shall need the point-like mass energy-momentum tensor**

$$T^{\mu\nu}(x) = M\int ds\,\delta\,(x - y(s))\,u^\mu u^\nu$$

**and also the matrix**

$$M^{\mu\nu,\alpha\beta}(x) = \frac{M}{4}\int ds\,\delta\,(x - y(s))\,u^\mu u^\nu u^\alpha u^\beta.$$

**The first step is the derivation of the one loop divergences in the theory of quantum gravity couples to classical material sources.**

$$\bar{\Gamma}^{(1)} = \frac{i}{2}\,\mathrm{Tr}\,\ln \hat{H} - i\,\mathrm{Tr}\,\ln \hat{H}_{gh},$$

**where**

$$\big(S + S_{gf}\big)^{(2)} = \frac{1}{2}\int d^4x\sqrt{-g}\,\phi_{\mu\nu}\,H^{\mu\nu,\alpha\beta}\,\phi_{\alpha\beta},$$

**and the bilinear form has an extra element $M^{\mu\nu,\alpha\beta}$ compared to the pure gravity**

$$H^{\mu\nu,\alpha\beta} = -\,(K^{\mu\nu,\alpha\beta}\Box + \Pi^{\mu\nu,\alpha\beta} + M^{\mu\nu,\alpha\beta}).$$

**We can use the relations**

$$g_{\mu\nu}M^{\mu\nu,\alpha\beta} = \frac{1}{4}T^{\alpha\beta}, \quad M_{\mu\nu,\alpha\beta}M^{\mu\nu,\alpha\beta} = \frac{1}{16}\,T_{\mu\nu}T^{\mu\nu},$$

$$T_{\mu\nu}T^{\mu\nu} = T^2, \qquad \textbf{and} \qquad T \equiv T^{\mu}_{\mu} = M\int ds\,\delta\,(x - y(s))\ .$$

**The calculations can be performed in a relatively simple way using minimal gauge fixing.**

**The final expression for the one loop divergences is**

$$\bar{\Gamma}^{(1)}_{div} = -\frac{\mu^{n-4}}{n-4}\int d^nx\sqrt{-g}\,\Big\{\beta_2 C^2 - \frac{1}{3}\beta_0 R^2 - 2\kappa^2\beta_{RT1}R_{\mu\nu}T^{\mu\nu} + \kappa^2\beta_{RT2}RT + \kappa^4\beta_{TT}T^2\Big\},$$

**where**

$$\beta_2 = \frac{4c_1+c_2}{2(4\pi)^2},\qquad \beta_0 = -\frac{c_1+c_2+3c_3}{(4\pi)^2},$$

$$\beta_{RT1} = -\frac{c_4}{2(4\pi)^2},\qquad \beta_{RT2} = \frac{c_5}{(4\pi)^2},\qquad \beta_{TT} = \frac{c_6}{(4\pi)^2}$$

**and**

$$c_1 = \frac{53}{45},\quad c_2 = -\frac{361}{90} - \frac{4\xi_2}{\gamma_1^4 Z^2},\quad c_3 = \frac{43}{36} + \frac{\xi_3}{3\gamma_1^4 Z^4},\quad c_4 = -\frac{2\xi_4}{\gamma_1^4 Z^4},$$

$$c_5 = \frac{5}{24} - \frac{\xi_5}{6\gamma_1^4 Z^4},\qquad c_6 = \frac{1}{32} - \frac{\xi_6}{4\gamma_1^4 Z^4},\qquad Z = \gamma_1 + 4\gamma_2 + 8\gamma_0\lambda\,.$$

**Clearly, there is a complicated parametrization dependence.**

**So, the off shell expression is**

$$\bar{\Gamma}^{(1)}_{div} = -\frac{\mu^{n-4}}{n-4}\int d^n x\sqrt{-g}\,\Big\{\beta_2 C^2 - \frac{1}{3}\beta_0 R^2 - 2\kappa^2\beta_{RT1}R_{\mu\nu}T^{\mu\nu} + \kappa^2\beta_{RT2}RT + \kappa^4\beta_{TT}T^2\Big\}.$$

**Using the classical equations of motion**

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{\kappa^2}{2}T_{\mu\nu},$$

**the divergences boil down to**

$$\bar{\Gamma}^{(1)}_{div}\Big|_{\text{on-shell}} = -\frac{\mu^{n-4}}{(4\pi)^2(n-4)}\int d^n x\sqrt{-g}\,\Big\{\frac{53}{45}R^2_{\mu\nu\alpha\beta} - \frac{373}{480}\,\kappa^4 T^2\Big\}$$

**independently of the choice of parametrization and gauge parameters, exactly as it should be.**

**However, we need the off shell result for our calculation.**

**Reconstruction of the finite part of effective action can be performed by using the rule valid for the massless fields contributions, such as the graviton**

$$\frac{\mu^{n-4}}{n-4} \quad \longrightarrow \quad \frac{1}{2}\,\ln\Big(\frac{\Box}{\mu^2}\Big)$$

**and we get, using the off shell expression,**

$$\bar{\Gamma}^{(1)} = -\int d^4x\sqrt{-g}\,\Big\{\frac{1}{2}\beta_2 C_{\mu\nu\alpha\beta}\ln\Big(\frac{\Box}{\mu^2}\Big)C^{\mu\nu\alpha\beta} - \frac{1}{6}\beta_0 R\ln\Big(\frac{\Box}{\mu^2}\Big)R$$
$$-\kappa^2\beta_{RT1}R_{\mu\nu}\ln\Big(\frac{\Box}{\mu^2}\Big)T^{\mu\nu} + \frac{\kappa^2}{2}\beta_{RT2}R\ln\Big(\frac{\Box}{\mu^2}\Big)T + \frac{\kappa^4}{2}\beta_{TT}T\ln\Big(\frac{\Box}{\mu^2}\Big)T\Big\}.$$

**The next steps are needed to eliminate gravity and arrive at the effective interaction between two classical massive sources.**

**Indeed, the gauge-fixing and parametrization independence represents a kind of a perfect testing for the procedure.**

# General formalism without effective approach

**In the weak-field regime, we can safely restrict consideration by the second order in curvature terms.**

**Then the most general action of gravity is**

$$S_{gr} = c_0 \int R + \int R_{\mu\nu\alpha\beta} \tilde{F}_1(\Box) R^{\mu\nu\alpha\beta} + \int R_{\mu\nu} \tilde{F}_2(\Box) R^{\mu\nu} + \int R \tilde{F}_3(\Box) R, \qquad \int \equiv \int d^4x \sqrt{-g}.$$

**One can prove that**

$$R_{\mu\nu\alpha\beta} F(\Box) R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} F(\Box) R^{\mu\nu} + R F(\Box) R = \mathcal{O}(R^3_{\dots}),$$

**reducing the action to**

$$S_{gr} = c_0 \int R + \int R_{\mu\nu} F_1(\Box) R^{\mu\nu} + \int R F_2(\Box) R.$$

$$S_{gr} = c_0 \int R + \int R_{\mu\nu} F_1(\Box) R^{\mu\nu} + \int R F_3(\Box) R.$$

**Using the known expansions and gauge invariance, it is possible to reduce this expression to the equations of motion**

$$\begin{aligned}\varepsilon^{\mu\nu} &= \frac{c_0}{2}\Big\{ a_1(\Box)\big(\Box h^{\mu\nu} - \partial^\mu\partial^\lambda h^\nu_\lambda - \partial^\nu\partial^\lambda h^\mu_\lambda\big) \\ &+ a_2(\Box)\big(\eta^{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} - \eta^{\mu\nu}\Box h + \partial^\mu\partial^\nu h\big) \\ &+ \big[a_1(Box) - a_2(\Box)\big]\,\frac{1}{\Box}\,\partial^\mu\partial^\nu\partial^\alpha\partial^\beta h_{\alpha\beta} = \frac{1}{2}\,T^{\mu\nu},\end{aligned}$$

**where**

$$a_1(\Box) = 1 + \frac{1}{c_0}\,F_1(\Box)\,, \qquad a_2(\Box) = 1 - \frac{1}{c_0}\,\big[F_1(\Box) + 4F_2(\Box)\big]\Box\,.$$

**Consider the spherically symmetric static metric**

$$\begin{aligned}&ds^2 = (1+2\varphi)dt^2 - (1-2\psi)(dx^2+dy^2+dz^2), \\ &\varphi = \varphi(r), \quad \psi = \psi(r), \quad r^2 = x^2+y^2+z^2, \quad \Box \longrightarrow -\Delta\,.\end{aligned}$$

The equations boil down to

$$\big[a_1(-\Delta) - a_2(-\Delta)\big]\,\Delta\varphi + a_2(-\Delta)\Delta\psi = -\frac{1}{2c_0}\,\rho,$$
$$\big[a_1(-\Delta) - 3a_2(-\Delta)\big]\,\Delta(\varphi - 2\psi) = -\frac{1}{2c_0}\,\rho.$$

These equations can be analysed for any particular choices of $a_1$ and $a_2$ or, equivalently, $F_1$ and $F_2$.

An interesting observation is that for GR, $\varphi = \psi$.

The same feature holds in the case $2F_1 + F_2 = 0$. Then $\varphi = \psi$. The action can be written in a special form

$$S_{gr} \,=\, \int \Big\{c_0 R \,+\, G_{\mu\nu}F_1(\Box)R^{\mu\nu}\Big\}.$$

This choice simplifies calculations and results, but from the physical viewpoint, there is nothing special in it.

**Using Fourier analysis, one can derive the following presentation (not too easy)**

$$\varphi(r) = -\frac{GM}{2\pi^2}\int \frac{d^3k}{k^2}\left[\frac{4}{3f_2(k^2)} + \frac{1}{3f_0(k^2)}\right] e^{i\vec{k}\cdot\vec{r}},$$
$$\psi(r) = -\frac{GM}{2\pi^2}\int \frac{d^3k}{k^2}\left[\frac{2}{3f_2(k^2)} + \frac{1}{3f_0(k^2)}\right] e^{i\vec{k}\cdot\vec{r}},$$

**where**

$$f_2(k^2) = a_1(k^2) \qquad f_0(k^2) = \frac{3}{2}\,a_2(k^2) - \frac{1}{2}\,a_1(k^2)\,.$$

**More elaborated form is**

$$\varphi(r) = -\frac{2GM}{\pi\, r}\int\limits_0^\infty dk\,\frac{\sin kr}{k}\left[\frac{4}{3f_2(k^2)} + \frac{1}{3f_0(k^2)}\right],$$
$$\psi(r) = -\frac{2GM}{\pi\, r}\int\limits_0^\infty dk\,\frac{\sin kr}{k}\left[\frac{2}{3f_2(k^2)} + \frac{1}{3f_0(k^2)}\right].$$

**After some calculus, these formulas give**

**I. GR.**

$$\varphi(r) = \psi(r) = -\frac{GM}{r}.$$

**II. Fourth derivative gravity**

$$\varphi(r) = -\frac{GM}{r}\left[1 + \frac{4}{3}e^{-m_2 r} - \frac{1}{3}e^{-m_0 r}\right],$$
$$\psi(r) = -\frac{GM}{r}\left[1 - \frac{2}{3}e^{-m_2 r} - \frac{1}{3}e^{-m_0 r}\right].$$

**The Newtonian singularity at the point $r = 0$ is removed by the effect of massive spin-zero and spin-two modes.**

**III. Superrenormalizable models with six and more derivatives**

**Newtonian singularity at the point $r = 0$ is washed away, independent on whether the theory is polynomial or nonlocal.**

**What changes if we apply the effective approach? Everyhing!**

**Then, all terms except the GR part are just perturbations.**

*J.Z. Simon, PRD 41 (1990); L. Parker & J.Z. Simon, PRD 47 (1993).*

**Consider small perturbations of the background metric around the Minkowski spacetime,**

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \qquad\qquad |\kappa h_{\mu\nu}| \ll 1$$

**Then**

$$\Gamma = -\frac{1}{2}\int d^4x \Big\{ \frac{1}{2} h_{\mu\nu} f_2 \Box h^{\mu\nu} - h^{\mu\nu} f_2 \partial_\mu \partial_\lambda h^\lambda_\nu - \frac{1}{6}\, h\,[f_2 + 2f_0]\,\Box h$$
$$+ \frac{1}{3} h\,[f_2 + 2f_0]\,\partial_\mu\partial_\nu h^{\mu\nu} + \frac{1}{3} h_{\alpha\beta}[f_2 - f_0]\frac{\partial^\alpha\partial^\beta\partial^\mu\partial^\nu}{\Box} h_{\mu\nu}\Big\}$$
$$-\frac{\kappa}{2}\int d^4x\,\Big\{ h_{\mu\nu} f_{RT1} T^{\mu\nu} + \kappa^2 \beta_{RT2}\, h_{\mu\nu} \ln\Big(\frac{\Box}{\mu^2}\Big)(\partial^\mu\partial^\nu - \eta^{\mu\nu}\Box) T \Big\},$$

**The form factors are**

$$f_i \;=\; f_i(\Box) \;=\; 1 + \kappa^2 \beta_i \ln\Big(\frac{\Box}{\mu^2}\Big)\Box, \qquad\qquad i = 2, 0, RT1.$$

**The term proportional to $\beta_{TT}$ has to be discarded, as it is $\mathcal{O}(M^2)$ and is beyond the weak-field approximation.**

**The perturbations in the isotropic static Newtonian form are**

$$\kappa h_{00} = 2\varphi(r), \qquad \kappa h_{ij} = 2\delta_{ij}\psi(r) \qquad \Box \longrightarrow -\Delta,$$

**and the source of the gravitational potentials should be a static point-like mass located at the origin,**

$$u^\mu = \delta^\mu_0, \qquad T^{\mu\nu} = \delta^\mu_0\delta^\nu_0\,\rho, \qquad \rho = M\delta(\vec{r}).$$

**The metric potentials can be obtained from the 00–component and from the trace of EoM,**

$$\begin{aligned}\big[f_2(-\Delta) - f_0(-\Delta)\big]\Delta\varphi + \big[f_2(-\Delta) + 2f_0(-\Delta)\big]\Delta\psi = \\ = \frac{3\kappa^2}{4}\Big[f_{RT1}(-\Delta) + \kappa^2\beta_{RT2}\ln\Big(-\frac{\Delta}{\mu^2}\Big)\Delta\Big]\rho, \qquad (1)\end{aligned}$$

$$f_0(-\Delta)(\Delta\varphi - 2\Delta\psi) = -\frac{\kappa^2}{4}\left[f_{RT1}(-\Delta) + 3\kappa^2\beta_{RT2}\ln(-\Delta/\mu^2)\right]\rho. \qquad (2)$$

**These eqs. represent the basis of the subsequent calculations.**

**Performing the loop expansion of the potentials we get,**

$$\varphi = \varphi^{(0)} + \varphi^{(1)} + \mathcal{O}(\hbar^2) \qquad \textbf{and} \qquad \psi = \psi^{(0)} + \psi^{(1)} + \mathcal{O}(\hbar^2).$$

**Since** $\beta_i = \mathcal{O}(\hbar)$, **the equations at zero and first orders are**

$$\begin{aligned}
\Delta\varphi^{(0)} &= \Delta\psi^{(0)} = \frac{\kappa^2 M}{4}\delta(\vec{r}), \\
\Delta\psi^{(1)} &= \frac{\kappa^2\beta_2}{3}\ln\Big(-\frac{\Delta}{\mu^2}\Big)\Delta^2\big[\varphi^{(0)} + \psi^{(0)}\big] \\
&\quad - \frac{\kappa^2\beta_0}{3}\ln\Big(-\frac{\Delta}{\mu^2}\Big)\Delta^2\big[\varphi^{(0)} - 2\psi^{(0)}\big] \\
&\quad + \frac{\kappa^4 M}{4}(-\beta_{RT1} + \beta_{RT2})\ln\Big(-\frac{\Delta}{\mu^2}\Big)\Delta\delta(\vec{r}), \\
\Delta\varphi^{(1)} - 2\Delta\psi^{(1)} &= \kappa^2\beta_0\ln\Big(-\frac{\Delta}{\mu^2}\Big)\Delta^2(\varphi^{(0)} - 2\psi^{(0)}) \\
&\quad + \frac{\kappa^4 M}{4}(\beta_{RT1} - 3\beta_{RT2})\ln\Big(-\frac{\Delta}{\mu^2}\Big)\Delta\delta(\vec{r}).
\end{aligned}$$

**These equations are linear and can be solved in a standard way.**

**In the three-dimensional Fourier space, we obtain the following solutions for the transformed potentials:**

$$\varphi^{(0)}(k) = \psi^{(0)}(k) = -\frac{\kappa^2 M}{4k^2},$$
$$\varphi^{(1)}(k) = \frac{\kappa^4 M}{4}\Big(\frac{4}{3}\beta_2 - \frac{1}{3}\beta_0 - \beta_{RT1} - \beta_{RT2}\Big)\ln\Big(\frac{k^2}{\mu^2}\Big),$$
$$\psi^{(1)}(k) = \frac{\kappa^4 M}{4}\Big(\frac{2}{3}\beta_2 + \frac{1}{3}\beta_0 - \beta_{RT1} + \beta_{RT2}\Big)\ln\Big(\frac{k^2}{\mu^2}\Big),$$

**where** $k = |\vec{k}|$. **Using the inverse Fourier transforms,**

$$\int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{r}}\frac{1}{k^2} = \frac{1}{4\pi r}, \qquad \int \frac{d^3k}{(2\pi)^3}\, e^{-i\vec{k}\cdot\vec{r}}\ln\Big(\frac{k^2}{\mu^2}\Big) = -\frac{1}{2\pi r^3},$$

**we arrive at the results**

$$\varphi(r) = -\frac{\kappa^2 M}{16\pi r} - \Big(\frac{4}{3}\beta_2 - \frac{1}{3}\beta_0 - \beta_{RT1} - \beta_{RT2}\Big)\frac{\kappa^4 M}{8\pi r^3},$$
$$\psi(r) = -\frac{\kappa^2 M}{16\pi r} - \Big(\frac{2}{3}\beta_2 + \frac{1}{3}\beta_0 - \beta_{RT1} + \beta_{RT2}\Big)\frac{\kappa^4 M}{8\pi r^3}.$$

**Once again,**

$$\varphi(r) = -\frac{\kappa^2 M}{16\pi r} - \left(\frac{4}{3}\beta_2 - \frac{1}{3}\beta_0 - \beta_{RT1} - \beta_{RT2}\right)\frac{\kappa^4 M}{8\pi r^3},$$

$$\psi(r) = -\frac{\kappa^2 M}{16\pi r} - \left(\frac{2}{3}\beta_2 + \frac{1}{3}\beta_0 - \beta_{RT1} + \beta_{RT2}\right)\frac{\kappa^4 M}{8\pi r^3}.$$

**In the standard $g_{\mu\nu} + \phi_{\mu\nu}$ parametrization limit, with $\gamma_1 \to 1$, $\gamma_{2,\ldots,6} \to 0$ and $\gamma_0 \to 0$. Then, we get**

$$\varphi(r) = -\frac{GM}{r}\left(1 + \frac{61}{60}\,\frac{G}{\pi r^2}\right),$$

$$\psi(r) = -\frac{GM}{r}\left(1 + \frac{23}{60}\,\frac{G}{\pi r^2}\right).$$

**The bad part is that these quantum corrections to the gravitational potentials are parametrization (and also gauge-fixing) dependent, therefore they do not directly correspond to physical observables.**

# The motion of a test particle

**Even though the loop corrections to the gravitational field generated by a point-like mass depend on parametrization and gauge fixing, physical observables should be invariant.**

**Consider the acceleration of a test particle moving in the gravitational field. The test particle couples with the quantum metric and its geodesic equation receives quantum corrections.**

**The two types of quantum corrections should combine into the invariant quantum corrections to Newton's law,** $m\vec{a} = \vec{F}$**.**

**The action of the test particle is**

$$S_m = -m \int \sqrt{g_{\mu\nu}\, dx^\mu dx^\nu}, \qquad x^\mu = (t, \vec{r}).$$

**We assume** $m \ll M$**, such that we can neglect contribution of the small mass to the potentials** $\varphi(r)$ **and** $\psi(r)$**.**

**The one-loop corrections can be obtained by the substitution $T^{\mu\nu} \longrightarrow T^{\mu\nu} + T_m^{\mu\nu}$, where**

$$T_m^{\mu\nu} = m \int ds\, \delta\,(y - x(s))\, u^\mu u^\nu . \tag{3}$$

**Applying this procedure, yields relevant terms**

$$\bar{\Gamma}_m^{(1)} = \int d^4x \sqrt{-g}\,\Big\{\kappa^2 \beta_{RT1} R_{\mu\nu} \ln\Big(\frac{\Box}{\mu^2}\Big) T_m^{\mu\nu} - \frac{1}{2}\kappa^2 \beta_{RT2} R \ln\Big(\frac{\Box}{\mu^2}\Big) T_m \,-\, \kappa^4 \beta_{TT} T \ln\Big(\frac{\Box}{\mu^2}\Big) T_m \Big\}.$$

**The total action for the test particle is $\Gamma_m = S_m + \bar{\Gamma}_m^{(1)}$. Taking the functional derivative with respect to $x_\mu$, we find**

$$\frac{1}{m}\,\frac{\delta \Gamma_m}{\delta x_\mu} = -\,\Big(\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\Big) + \frac{1}{m}\,\frac{\delta \bar{\Gamma}_m^{(1)}}{\delta x_\mu} = 0. \tag{4}$$

**As we already mentioned above,** $\ln\left(-\frac{\Delta}{\mu^2}\right)\delta(\vec{r}) = -\frac{1}{2\pi r^3}$**, that gives, after a small calculation**

$$\frac{1}{m}\frac{\delta\bar{\Gamma}_m^{(1)}}{\delta\vec{r}} = -\frac{\kappa^4 M}{8\pi}(-\beta_{RT1} - \beta_{RT2} + 4\beta_{TT})\vec{\nabla}\left(\frac{1}{r^3}\right). \tag{5}$$

**In our case, this gives**

$$\vec{a} = -\vec{\nabla}\left[-\frac{\kappa^2 M}{16\pi r} - \left(\frac{4}{3}\beta_2 - \frac{1}{3}\beta_0 - 2\beta_{RT1} - 2\beta_{RT2} + 4\beta_{TT}\right)\frac{\kappa^4 M}{8\pi r^3}\right].$$

**The quantum corrected Newtonian potential is defined from** $\vec{a} = -\mathrm{grad}U$. **Thus, we get**

$$U(r) = -\frac{\kappa^2 M}{16\pi r} - \left(\frac{4}{3}\beta_2 - \frac{1}{3}\beta_0 - 2\beta_{RT1} - 2\beta_{RT2} + 4\beta_{TT}\right)\frac{\kappa^4 M}{8\pi r^3}.$$

**and, finally,**

$$U(r) = -\frac{GM}{r}\left(1 + \frac{17}{20}\frac{G}{\pi r^2}\right),$$

**which does not depend on parametrization or gauge parameters.**

# Gauge invariance in quantum gravity (QG)

**Let us prove that the combination**

$$\beta_{\rm inv} = \frac{4}{3}\beta_2 - \frac{1}{3}\beta_0 - 2\beta_{RT1} - 2\beta_{RT2} + 4\beta_{TT}\,,$$

**is the gauge and parametrization invariant in effective QG.**

**We use the general statement about the gauge-fixing and parametrization independence of the on-shell effective action.**

**The difference between the divergences of two versions of the one-loop effective action, evaluated using different gauge and parametrization parameters $\alpha_i$ and $\alpha_0$ is proportional to the classical equations of motion**

$$\delta\bar{\Gamma}^{(1)}_{div} \,=\, \bar{\Gamma}^{(1)}_{div}(\alpha_i) - \bar{\Gamma}^{(1)}_{div}(\alpha_0) = -\frac{\mu^{n-4}}{(4\pi)^2(n-4)}\int d^nx\sqrt{-g}\,\varepsilon^{\mu\nu}f_{\mu\nu},$$

**where** $$\varepsilon^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R - \frac{\kappa^2}{2}T^{\mu\nu}.$$

**Remember** $$\bar{\Gamma}^{(1)}_{div} = -\frac{\mu^{n-4}}{n-4}\int d^n x\sqrt{-g}\,\Big\{\beta_2 C^2 - \frac{1}{3}\beta_0 R^2 - 2\kappa^2\beta_{RT1}R_{\mu\nu}T^{\mu\nu} + \kappa^2\beta_{RT2}RT + \kappa^4\beta_{TT}T^2\Big\},$$

**Since divergences are local and covariant functionals of the mass dimension four, the tensor function $f_{\mu\nu}$ has the general structure**

$$f_{\mu\nu} = b_1 R_{\mu\nu} + b_2 R g_{\mu\nu} + \kappa^2 b_3 T_{\mu\nu} + \kappa^2 b_4 T g_{\mu\nu},$$

**where the parameters $b_{1,2,3,4}$ depend on the choice of gauge and parametrization parameters $\alpha_i$. Thus,**

$$\begin{aligned}\delta\bar{\Gamma}^{(1)}_{div} &= -\frac{\mu^{n-4}}{(4\pi)^2(n-4)}\int d^n x\sqrt{-g}\,\Big\{b_1 R^2_{\mu\nu} - \big(\tfrac{1}{2}b_1 + b_2\big)R^2 \\ &+ \kappa^2\big(b_3 - \tfrac{1}{2}b_1\big)R_{\mu\nu}T^{\mu\nu} - \kappa^2\big(\tfrac{1}{2}b_2 + \tfrac{1}{2}b_3 + b_4\big)RT \\ &- \kappa^4\big(\tfrac{1}{2}b_3 + \tfrac{1}{2}b_4\big)T^2\Big\}.\end{aligned}$$

**So, under the gauge and/or parametrization changes, the coefficients of divergences transform as**

$$c_1 \to c_1, \quad c_2 \to c_2 + b_1, \quad c_3 \to c_3 - \left(\frac{b_1}{2} + b_2\right) ,$$
$$c_4 \to c_4 + \left(b_3 - \frac{b_1}{2}\right) , \quad c_5 \to c_5 - \frac{1}{2}(b_2 + b_3 + 2b_4) ,$$
$$c_6 \to c_6 - \frac{1}{2}(b_3 + b_4) .$$

**It is easy to check that $c_1$ is invariant (this is knows from 70-th), and that there is the second independent combination**

$$c_{\rm inv} = c_2 + c_3 + c_4 - 2c_5 + 4c_6 .$$

**And this directly implies that**

$$\beta_{\rm inv} = \frac{4}{3}\beta_2 - \frac{1}{3}\beta_0 - 2\beta_{RT1} - 2\beta_{RT2} + 4\beta_{TT} ,$$

**is invariant under the gauge and parametrization changes in the QG based on GR.**

**We conclude that the method of**

[DM] *D.A.R. Dalvit and F.D. Mazzitelli, PRD; hep-th/9708102*

**provides a consistent scheme of calculating QG corrections to Newtonian potential in the effective framework.**

**This scheme is more complicated than the derivation of diagrams (amplitudes), but it excludes the quantization of the classical source, e.g., astrophysical macroscopic objects.**

**The main feature of the scheme [DM97] is the use of geodesics as an ultimate element of setting the result on shell.**

**The implementation of effective approach is that the IR quantum corrections to GR are treated as small perturbations. In case loop corrections are treated at the same level as the GR contribution, the result would be dramatically different.**

# Conclusions

- The effective QG is our simplest option for the QG.

- Assuming effective QG means we give up from the main target of QG, i.e., from the describing QG in the deep UV, explanation (or removal) of singularities and alike.

- The “standard” object of interest in the effective QG is the interaction between two macroscopic bodies. The consistent approach requires discarding the diagrams with external lines of the macroscopic sources.

- In this reduced setting, achieving consistent results is more complicated than usual in QFT owing to the presence of external classical sources. The problem was solved in [DM97] by going on shell for the massive test particle.

- The numerical effect for the two static bodies is negligible, but still represents a great interest as an example of a consistent calculation of final observables in QG.